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Chebyshev Expansion Methods for the Solution of the Extended Graetz Problem

HWAR-CHING KU AND DIMITRI HATZIAVRAMIDIS

Department of Chemical Engineering, Illinois Institute of Technology, Chicago, Illinois 60616

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The extended Graetz problem, a classical problem described by an equation with a non-selfadjoint second-order elliptic differential operator, is solved by two numerical methods based on Chebyshev expansions: A Chebyshev-finite difference method and a Chebyshev-"finite" element method. The latter relies entirely on Chebyshev expansions, global in the bounded (radial) coordinate direction and local in the unbounded (axial) direction and proves to be more accurate than the Chebyshev-finite difference method in resolving singularities. Both methods can be applied to general type boundary-value problems involving second-order elliptic operators, have accuracy comparable to high order finite difference schemes and are operation cost efficient. © 1984 Academic Press, Inc.

INTRODUCTION

In earlier papers [1, 2] Chebyshev expansion techniques for the solution of initial-value laminar forced convection problems in pipes were presented. Those problems, with axial conduction neglected, exhibit a boundary layer structure. With axial conduction included (low Peclet number), heat transfer problems in laminar pipe flow constitute boundary-value problems of the elliptic type. When the physical properties of the system involved are taken to be constant, the extended Graetz problem is defined. This problem has long been in the midst of numerous attempts, analytical [3–5] and computational [6–8], to generate methods for the solution of second-order elliptic differential equations involving non-selfadjoint operators.

Because of their fast convergence (exponential convergence as compared to algebraic of finite difference solvers) and the capability to accurately resolve thin layers where steep changes in the field variables (and/or the transport coefficients) occur, Chebyshev expansion methods are particularly suitable for the solution of the extended Graetz problem. Chebyshev expansion methods were used for the solution of Poisson's equation in a rectangle with Dirichlet and/or periodic boundary conditions by Haidvogel and Zang [9], Delves and Hall [10], and Orszag [11]. The applicability of these methods to the extended Graetz problem is limited by their specificity (solvers for Poisson's equation with boundary conditions of certain types),

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the absence of a mechanism to treat singularities [9] or by poor computational economy features [11].

We have developed two Chebyshev expansion methods for the solution of the extended Graetz problem which can be applied to the solution of general type elliptic equations with no restrictions on the boundary conditions and which are easily implemented and operation cost efficient. Before we proceed with the description of the methods as applied to the extended Graetz problem, we present a summary of the methods used for the calculation of derivatives and integrals from Chebyshev expansions.

Calculation of Derivatives and Integrals from Chebyshev Expansions

For a function $f(x)$ with $x \in [-1, 1]$, the $N+1$-term Chebyshev expansion is

$$f(x) = \sum_{p=1}^{N+1} a_p T_{p-1}(x). \tag{1}$$

With the collocation points selected as $x_n = \cos(\pi(n-1)/N)$ $(1 \leqslant n \leqslant N+1)$, the expansion (1) becomes

$$f_n = f(x_n) = \sum_{p=1}^{N+1} a_p \cos\frac{\pi(p-1)(n-1)}{N} \tag{2a}$$

with a_p given by

$$a_p = \frac{2}{N}\frac{1}{\bar{c}_p}\sum_{n=1}^{N+1} \bar{c}_n^{-1} T_{p-1}(x_n) f_n \tag{2b}$$

and $\bar{c}_1 = \bar{c}_{N+1} = 2$, $\bar{c}_p = 1$ for $1 < p < N+1$. In matrix form, Eq. (2b) can be written as

$$\mathbf{a} = \hat{\mathbf{T}}\mathbf{f} \tag{3}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \\ a_{N+1} \end{bmatrix}$$

$$\hat{\mathbf{T}} = \frac{2}{N}\begin{bmatrix} \frac{T_0(x_1)}{4} & \frac{T_0(x_2)}{2} & \cdots & \frac{T_0(x_N)}{2} & \frac{T_0(x_{N+1})}{4} \\ \frac{T_1(x_1)}{2} & T_1(x_2) & \cdots & T_1(x_N) & \frac{T_1(x_{N+1})}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{T_{N-1}(x_1)}{2} & T_{N-1}(x_2) & \cdots & T_{N-1}(x_N) & \frac{T_{N-1}(x_{N+1})}{2} \\ \frac{T_N(x_1)}{4} & \frac{T_N(x_2)}{2} & \cdots & \frac{T_N(x_N)}{2} & \frac{T_N(x_{N+1})}{4} \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \\ f_{N+1} \end{bmatrix}.$$

The first derivative of $f(x)$ can be approximated as

$$f'(x_n) = \sum_{p=1}^{N+1} a_p^{(1)} T_{p-1}(x_n) \tag{4a}$$

where $a_p^{(1)}$ are now given by

$$a_p^{(1)} = \frac{2}{c_p} \sum_{\substack{n=p+1 \\ n+p \text{ odd}}}^{N+1} (n-1)\, a_n \tag{4b}$$

and

$$c_1 = 2, \qquad c_p = 1, \qquad p \geqslant 2.$$

Equations (4a) and (4b), in matrix form are written as

$$\mathbf{f}' = \mathbf{T}\mathbf{a}^{(1)} \tag{5a}$$

and

$$\mathbf{a}^{(1)} = \mathbf{G}^{(1)}\mathbf{a} \tag{5b}$$

respectively, where

$$\mathbf{T} = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_N(x_2) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ T_0(x_N) & T_1(x_N) & \cdots & T_N(x_N) \\ T_0(x_{N+1}) & T_1(x_{N+1}) & \cdots & T_N(x_{N+1}) \end{bmatrix}$$

and $\mathbf{G}^{(1)}$ is a $(N+1) \times (N+1)$ matrix with elements

$$\begin{aligned} G_{ij}^{(1)} &= 0 && \text{if} \quad i \geqslant j \text{ or } i+j \text{ even} \\ &= \frac{2(j-1)}{c_i} && \text{otherwise.} \end{aligned} \tag{6}$$

Combining Eqs. (5a), (5b) and (3), we get

$$\mathbf{f}' = \mathbf{T}\mathbf{G}^{(1)}\mathbf{a} = \mathbf{T}\mathbf{G}^{(1)}\hat{\mathbf{T}}\mathbf{f} = \hat{\mathbf{G}}^{(1)}\mathbf{f} \tag{7a}$$

where

$$\hat{\mathbf{G}}^{(1)} = \mathbf{T}\mathbf{G}^{(1)}\hat{\mathbf{T}}. \tag{7b}$$

In a similar way, the approximation to the second derivative is

$$f''(x_n) = \sum_{p=1}^{N+1} a_p^{(2)} T_{p-1}(x_n) \tag{8a}$$

with

$$a_p^{(2)} = \frac{2}{c_p} \sum_{\substack{n=p+1 \\ n+p \text{ odd}}}^{N+1} (n-1)\, a_n^{(1)}. \tag{8b}$$

In matrix form, Eqs. (8a) and (8b) can be written as

$$\mathbf{f}'' = \mathbf{T}\mathbf{a}^{(2)} \tag{9a}$$

and

$$\mathbf{a}^{(2)} = \mathbf{G}^{(1)}\mathbf{a}^{(1)} \tag{9b}$$

respectively. Equations (9a) and (9b), in view of Eqs. (5a) and (5b), give

$$\mathbf{a}^{(2)} = \mathbf{G}^{(1)}\mathbf{a}^{(1)} = \mathbf{G}^{(1)}\mathbf{G}^{(1)}\mathbf{a} = \mathbf{G}^{(2)}\mathbf{a} \tag{10a}$$

and

$$\mathbf{f}'' = \mathbf{T}\mathbf{G}^{(2)}\mathbf{a} = \mathbf{T}\mathbf{G}^{(2)}\hat{\mathbf{T}}\mathbf{f} = \hat{\mathbf{G}}^{(2)}\mathbf{f} \tag{10b}$$

where

$$\mathbf{G}^{(2)} = \mathbf{G}^{(1)}\mathbf{G}^{(1)} \tag{11}$$

and

$$\hat{\mathbf{G}}^{(2)} = \mathbf{T}\mathbf{G}^{(2)}\hat{\mathbf{T}}. \tag{12}$$

This completes the calculation of the derivatives of a function from its Chebyshev expansion.

Regarding integrals of $f(x)$, the integral $\int_{-1}^{x} f(\hat{x})\, d\hat{x}$ for $f(x)$ being sufficiently smooth, can be calculated from [12]

$$\int_{-1}^{x} f(\hat{x})\, d\hat{x} = \sum_{p=1}^{N+2} b_p T_{p-1}(x) \tag{13}$$

with $b_{N+2} = a_{N+1}/2(N+1)$, $b_{N+1} = a_N/2N$, $b_n = (1/2(n-1))(c_{n-1}a_{n-1} - a_{n+1})$, $n = 2, 3, \ldots, N$ (a_n are the coefficients of expansion (1)). By requiring that the integral vanishes for $x = -1$, we obtain for $\int_{-1}^{1} f(x)\, dx$

$$\begin{aligned}\int_{-1}^{1} f(x)\, dx &= 2(b_2 + b_4 + \cdots)\\ &= 2\left(a_1 - \frac{a_3}{1 \cdot 3} - \cdots\right)\\ &= 2a_1 - \sum_{\substack{p=2\\ p=\text{even}}}^{N} \left[\frac{2}{p^2 - 1}\right] a_{p+1}\\ &= \mathbf{I}^T\mathbf{a}\end{aligned} \tag{14}$$

where

$$\mathbf{I}^T = 2\left[1, 0, -\frac{1}{3}, 0, -\frac{1}{3 \cdot 5}, 0, \ldots\right].$$

Equation (14), in view of Eq. (3), can be written as

$$\int_{-1}^{1} f(x)\, dx = \mathbf{I}^T\hat{\mathbf{T}}\mathbf{f} = \mathbf{W}^T\mathbf{f} \tag{15}$$

with $\mathbf{W}^T = \mathbf{I}^T\hat{\mathbf{T}}$. This integration procedure was used to calculate bulk temperatures needed in the evaluation of the Nusselt numbers for the extended Graetz problem.

If the interval of definition, $[x_1, x_2]$, for the function under consideration is different than $[-1, 1]$, $[x_1, x_2]$ is mapped onto $[-1, 1]$ through

$$x' = \frac{2x - (x_1 + x_2)}{x_2 - x_1} \tag{16}$$

before the previously described procedures for calculating derivatives and integrals are applied.

The Extended Graetz Problem

The energy equation for hydrodynamically developed flow in a circular pipe, with axial conduction included, reads

$$\alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right)=u_z\frac{\partial T}{\partial z} \tag{17}$$

where α is the thermal diffusivity, r is the radial and z the axial distance, respectively, T is the local temperature and u_z follows a Poiseuille distribution. The retaining of the axial conduction term in the energy equation is a consequence of the low Peclet number, $\mathrm{Pe}=2R\langle u_z\rangle/\alpha$ (R is the radius of the pipe and $\langle u_z\rangle$ the cross-sectional area averaged velocity). We seek the solution of Eq. (17) in the case of a stepwise varying wall temperature. Equation (17) is solved subject to the boundary conditions

$$z=-\infty,\qquad 0\leqslant r\leqslant R,\qquad T=T_0 \tag{18a}$$

$$z=+\infty,\qquad 0\leqslant r\leqslant R,\qquad T=T_w \tag{18b}$$

$$z\leqslant 0,\qquad r=R,\qquad T=T_0 \tag{18c}$$

$$z>0,\qquad r=R,\qquad T=T_w \tag{18d}$$

$$-\infty<z<+\infty,\qquad r=0,\qquad \frac{\partial T}{\partial r}=0. \tag{18e}$$

The field variables and the radial distance are non-dimensionalized as

$$r'=\frac{r}{R},\qquad T'=\frac{T-T_w}{T_0-T_w},\qquad u_z'=\frac{u_z}{\langle u_z\rangle} \tag{19}$$

while the unbounded z-domain is mapped into a finite domain through the transformation advanced by Verhoff and Fisher [7]

$$z'=\frac{1}{\pi}\tan^{-1}\frac{z}{\kappa R}. \tag{20}$$

In Eq. (20), κ is a constant at our disposal which facilitates the amplification of the near $z'=0$ region, where steep changes in temperature occur. With this non-dimensionalization Eq. (17) becomes

$$\mathrm{Pe}(1-r'^2)\frac{\cos^2 \pi z'}{\pi \kappa}\frac{\partial T'}{\partial z'}$$
$$=\frac{\partial^2 T'}{\partial r'^2}+\frac{1}{r'}\frac{\partial T'}{\partial r'}+\frac{1}{(\pi\kappa)^2}\left[\cos^4 \pi z' \frac{\partial^2 T'}{\partial z'^2}-2\pi\cos^3 \pi z' \sin \pi z' \frac{\partial T'}{\partial z'}\right] \quad (21)$$

subject to the boundary conditions

$$z'=-0.5, \quad 0\leqslant r'\leqslant 1, \quad T'=1 \quad (22a)$$
$$z'=+0.5, \quad 0\leqslant r'\leqslant 1, \quad T'=0 \quad (22b)$$
$$z'\leqslant 0, \quad r'=1, \quad T'=1 \quad (22c)$$
$$z'>0, \quad r'=1, \quad T'=0 \quad (22d)$$
$$-0.5<z'<+0.5, \quad r'=0, \quad \frac{\partial T'}{\partial r'}=0. \quad (22e)$$

We present below two Chebyshev expansion methods for the solution of the extended Graetz problem. In both these methods Chebyshev expansions are used to evaluate the radial derivatives according to the matrix scheme earlier described. In the Chebyshev-finite difference method, finite difference approximations are used for the axial derivatives, while in the Chebyshev-"finite" element method, the z'-region is subdivided in a finite number of subregions and Chebyshev expansions confined to every subregion are used to calculate the axial derivatives.

Chebyshev-Finite Difference Method

The domain of solution, $(-0.5, +0.5)\otimes(0, 1)$, for the previously described problem is discretized to include $NZ+1$ equally spaced points in the z'-direction and $NR+1$ points in the radial direction. The latter are selected as the collocation points of the Chebyshev expansions in the radial direction. If second order finite difference approximations are used for the axial derivatives appearing in Eq. (21), while the radial derivatives in the same approximation are evaluated from Chebyshev expansions as earlier described, the discrete form of Eq. (21) is

$$a_{i,j}T'_{i-1,j}+b_{i,j}T'_{i,j}+c_{i,j}T'_{i+1,j}-\sum_{l=2}^{NR} T'_{i,l}d_{j,l}=g_{i,j} \quad (23)$$

with

$$a_{i,j}=-\mathrm{Pe}(1-r_j'^2)\frac{\cos^2 \pi z_i'}{\pi\kappa}\frac{1}{2\Delta z'}-\frac{\cos^4 \pi z_i'}{(\pi\kappa)^2}\frac{1}{\Delta z'^2}-\frac{\pi}{(\pi\kappa)^2}\frac{\cos^3 \pi z_i' \sin \pi z_i'}{\Delta z'} \quad (24a)$$

$$b_{i,j} = \frac{\cos^4 \pi z_i'}{(\pi\kappa)^2} \frac{2}{\Delta z'^2} \tag{24b}$$

$$c_{i,j} = \mathrm{Pe}(1 - r_j'^2) \frac{\cos^2 \pi z_i'}{\pi\kappa} \frac{1}{2\Delta z'} - \frac{\cos^4 \pi z_i'}{(\pi\kappa)^2} \frac{1}{\Delta z'^2} + \frac{\pi}{(\pi\kappa)^2} \frac{\cos^3 \pi z_i' \sin \pi z_i'}{\Delta z'} \tag{24c}$$

$$d_{j,l} = \hat{G}R_{j,l}^{(2)} + \frac{\hat{G}R_{j,l}^{(1)}}{r_j'} - \frac{\hat{G}R_{NR+1,l}^{(1)} \hat{G}R_{j,NR+1}^{(2)} + \hat{G}R_{NR+1,l}^{(1)} \hat{G}R_{j,NR+1}^{(1)}/r_j'}{\hat{G}R_{NR+1,NR+1}^{(1)}} \tag{24d}$$

$$g_{i,j} = \left[\hat{G}R_{j,1}^{(2)} + \frac{\hat{G}R_{j,1}^{(1)}}{r_j'} - \frac{\hat{G}R_{NR+1,1}^{(1)} \hat{G}R_{j,NR+1}^{(2)} + \hat{G}R_{NR+1,1}^{(1)} \hat{G}R_{j,NR+1}^{(1)}/r_j'}{\hat{G}R_{NR+1,NR+1}^{(1)}}\right] T_{i,1}' \tag{24e}$$

and $\hat{G}R_{i,j}^{(1)}$, $\hat{G}R_{i,j}^{(2)}$ are the ij-components of the $\hat{\mathbf{G}}^{(1)}$ and $\hat{\mathbf{G}}^{(2)}$ matrices defined in Eqs. (7b) and (12), respectively, from Chebyshev expansions in the r'-direction. The boundary conditions of Eq. (22e) can be implemented as

$$\sum_{l=1}^{NR+1} T_{i,l}' \hat{G}R_{NR+1,l}^{(1)} = 0. \tag{25}$$

Equation (23), together with the boundary conditions, can be recast in the form

$$\mathbf{AX} = \mathbf{B} \tag{26}$$

where $\mathbf{A}$ has the structure

$$\mathbf{A} = \begin{bmatrix} \mathbf{D}^{(1)}\mathbf{E}^{(1)} & & & \\ \mathbf{C}^{(2)}\mathbf{D}^{(2)}\mathbf{E}^{(2)} & & & \\ & \ddots & & \\ & & \mathbf{C}^{(NZ-2)}\mathbf{D}^{(NZ-2)}\mathbf{E}^{(NZ-2)} & \\ & & & \mathbf{C}^{(NZ-1)}\mathbf{D}^{(NZ-1)} \end{bmatrix}. \tag{27a}$$

$\mathbf{X}$ and $\mathbf{B}$ can be partitioned into

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \\ \vdots \\ \mathbf{X}^{(NZ-1)} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \\ \vdots \\ \mathbf{B}^{(NZ-1)} \end{bmatrix}. \tag{27b}$$

$\mathbf{C}^{(i)}$, $\mathbf{D}^{(i)}$ and $\mathbf{E}^{(i)}$, $i = 1,..., NZ - 1$, are $(NR - 1) \times (NR - 1)$ matrices with elements

$$\begin{aligned} C_{j,l}^{(i)} &= a_{i+1,j+1} \quad &&\text{for } j = l \\ &= 0 &&\text{otherwise} \end{aligned} \tag{28a}$$

$$D_{j,l}^{(i)} = b_{i+1,j+1} - d_{j+1,l+1} \quad \text{for} \quad j = l$$
$$= d_{j+1,l+1} \quad \text{otherwise} \tag{28b}$$

$$E_{j,l}^{(i)} = c_{i+1,j+1} \quad \text{for} \quad j = l$$
$$= 0 \quad \text{otherwise} \tag{28c}$$

The vectors $\mathbf{X}^{(i)}$ and $\mathbf{B}^{(i)}$ are given by

$$\mathbf{X}^{(i)} = \begin{bmatrix} T'_{i+1,2} \\ T'_{i+1,3} \\ \vdots \\ T'_{i+1,NR} \end{bmatrix}, \quad i = 1,..., NZ-1 \tag{29}$$

and

$$\mathbf{B}^{(1)} = \begin{bmatrix} g_{2,2} \\ g_{2,3} \\ \vdots \\ g_{2,NR} \end{bmatrix} - \mathbf{C}^{(1)} \begin{bmatrix} T'_{1,2} \\ T'_{1,3} \\ \vdots \\ T'_{1,NR} \end{bmatrix} \tag{30a}$$

$$\mathbf{B}^{(i)} = \begin{bmatrix} g_{i+1,2} \\ g_{i+1,3} \\ \vdots \\ g_{i+1,NR} \end{bmatrix}, \quad i = 2,..., NZ-2 \tag{30b}$$

$$\mathbf{B}^{(NZ-1)} = \begin{bmatrix} g_{NZ,2} \\ g_{NZ,3} \\ \vdots \\ g_{NZ,NR} \end{bmatrix} - \mathbf{E}^{(NZ-1)} \begin{bmatrix} T'_{NZ+1,2} \\ T'_{NZ+1,3} \\ \vdots \\ T'_{NZ+1,NR} \end{bmatrix}. \tag{30c}$$

Equation (26) can be solved by applying the LU decomposition [13, 14]. The storage required for $\mathbf{A}$, $(NR-1) \times (NR) \times (NZ-1)$, is less than the storage required by a straight finite difference method of comparable accuracy.

Chebyshev-"Finite" Element Method

This method is similar to the "global element" method of Delves and Hall [10]. It is different from the latter in the following:

1. In the global element method, local Chebyshev expansions are used in both

directions of a two dimensional domain. Here, local dimensions are used for the extended (axial) direction only.

2. In the global element method, the expansion coefficients are determined as stationary points of a variational functional. Here, the solution is obtained in terms of the field variables and not the expansion coefficients.

These two differences account for the superiority of the method described here in terms of computational efficiency and economy, when it is compared to the global element method. Both methods share generality in their applicability to second order elliptic problem with self-adjoint or non-self-adjoint operators and a variety of boundary conditions (homogeneous or nonhomogeneous). Both methods treat the differential equation and the boundary conditions on equal footing and can be shown to resolve efficiently singularities in the domain of solution or its boundaries (in the extended Graetz problem a singularity exists for the pipe wall temperature at $z' = 0$).

The use of a global Chebyshev expansion for the evaluation of the axial derivatives is accompanied by severe errors (Gibbs oscillations) which arise from steep changes in the temperature in the neighborhood of $z' = 0$. These Gibbs phenomena can not be smoothed out by filtering techniques [2]. The z'-domain, instead, is divided into a small number, NE, of "elements." Chebyshev expansions in the z'-direction are employed for each element while continuity of temperature and heat fluxes is imposed at the element–element interface. Thus the approximation to Eq. (21) in the interior of each element becomes

$$a_{i,j} \sum_{k=1}^{NZ+1} \hat{G}Z^{(1)}_{i,k} T'_{k,j} - b_{i,j} \sum_{k=1}^{NZ+1} \hat{G}Z^{(2)}_{i,k} T'_{k,j} - \sum_{l=2}^{NR} T'_{i,l} d_{j,l} = g_{i,j} \tag{31}$$

with

$$a_{i,j} = \mathrm{Pe}(1 - r_j'^2) \frac{\cos^2 \pi z_i'}{\pi \kappa} + \frac{2\pi}{(\pi\kappa)^2} \cos^3 \pi z_i' \sin \pi z_i' \tag{32a}$$

$$b_{i,j} = \frac{1}{(\pi\kappa)^2} \cos^4 \pi z_i' \tag{32b}$$

for $i = 1,\ldots, NZ + 1$ and $j = 2,\ldots, NR$ and $d_{j,l}, g_{i,j}$ given by (24d) and (24e), respectively. $\hat{G}Z^{(1)}_{i,k}$ and $\hat{G}Z^{(2)}_{i,k}$ are the ik-elements of the $\hat{\mathbf{G}}^{(1)}$ and $\hat{\mathbf{G}}^{(2)}$ matrices respectively, constructed from local Chebyshev expansions in the z'-direction. The continuity requirements at the element–element interface yield

$$T'_{NZ+1,j}\big|_{\text{element } N-1} = T'_{1,j}\big|_{\text{element } N} \tag{33a}$$

and

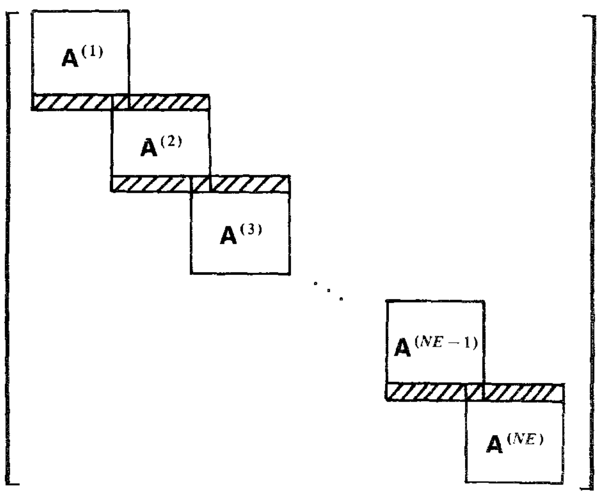
$$\frac{1}{L}\sum_{k=1}^{NZ+1} \hat{G}Z^{(1)}_{NZ+1,k}T'_{k,j}\Big|_{\text{element } N-1} = \frac{1}{L}\sum_{k=1}^{NZ+1} \hat{G}Z^{(1)}_{1,k}T'_{k,j}\Big|_{\text{element } N} \qquad (33b)$$

for $j = 2,..., NR$ and $N = 2,..., NE$. L is the length of the element. Equation (31) together with the continuity requirements (33a) and (33b) and the boundary conditions can be cast into the form

$$\mathbf{AX} = \mathbf{B} \qquad (34)$$

where $\mathbf{A}$ has the diagonal form

$\mathbf{A} =$



(35a)

and $\mathbf{X}$ and $\mathbf{B}$ can be partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \\ \vdots \\ \mathbf{X}^{(NE-1)} \\ \mathbf{X}^{(NE)} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \\ \vdots \\ \mathbf{B}^{(NE-1)} \\ \mathbf{B}^{(NE)} \end{bmatrix}. \qquad (35b)$$

The hatched area in Eq. (35a) arises from the coupling due to the continuity requirements at the element–element interface. The blocks $\mathbf{A}^{(N)}$, $N = 1,..., NE$, of size $(NZ + 1)^2 \times (NR - 1)^2$ with a $(NR - 1) \times (NR - 1)$ overlap region at the corners, consists of elements $\mathbf{C}^{(i,k)}$, $i, k = 1,..., NZ + 1$, which in turn are $(NR - 1) \times (NR - 1)$ matrices with elements $C^{(i,k)}_{j,l}$ $(j, l = 1,..., NR - 1)$ of the following form.

For $\mathbf{A}^{(1)}$:

$$C_{j,l}^{(1,1)} = 1 \quad \text{if} \quad j = l$$
$$= 0 \quad \text{otherwise} \tag{36a}$$

$$C_{j,l}^{(i,k)} = a_{j+1,i}\hat{G}Z_{i,k}^{(1)} - b_{j+1,i}\hat{G}Z_{i,k}^{(2)} \quad \text{if} \quad j = l \text{ and } i \neq k$$
$$= 0 \quad \text{if} \quad j \neq l \text{ and } i \neq k$$
$$= a_{j+1,i}\hat{G}Z_{i,k}^{(1)} - b_{j+1,i}\hat{G}Z_{i,k}^{(2)} - d_{j+1,l+1} \quad \text{if} \quad j = l \text{ and } i = k$$
$$= -d_{j+1,l+1} \quad \text{if} \quad j \neq l \text{ and } i = k \tag{36b}$$

$$C_{j,l}^{(NZ+1,k)} = \hat{G}Z_{NZ+1,k}^{(1)} \quad \text{if} \quad j = l \text{ and } k \neq Nz + 1$$
$$= \hat{G}Z_{NZ+1,NZ+1}^{(1)}\big|_{\text{element }1} - \hat{G}Z_{1,1}^{(1)}\big|_{\text{element }2} \quad \text{if} \quad j = l \text{ and } k = NZ + 1$$
$$= 0 \quad \text{if} \quad j \neq l \text{ and } k = NZ + 1. \tag{36c}$$

For $\mathbf{A}^{(N)}$ $(N = 2,..., NE - 1)$:

$$C_{j,l}^{(1,1)} = C_{j,l}^{(NZ+1,NZ+1)}\big|_{\text{element }N-1} \quad \text{if} \quad j = l$$
$$= 0 \quad \text{otherwise} \tag{37a}$$

$$C_{j,l}^{(1,k)} = -\hat{G}Z_{1,k}^{(1)} \quad \text{if} \quad j = l$$
$$= 0 \quad \text{otherwise } (k \neq 1) \tag{37b}$$

$$C_{j,l}^{(i,k)} = a_{j+1,i}\hat{G}Z_{i,k}^{(1)} - b_{j+1,i}\hat{G}Z_{i,k}^{(2)} \quad \text{if} \quad j = l \text{ and } i \neq k$$
$$= 0 \quad \text{if} \quad j \neq l \text{ and } i \neq k$$
$$= a_{j+1,i}\hat{G}Z_{i,k}^{(1)} - b_{j+1,i}\hat{G}Z_{i,k}^{(2)} - d_{j+1,l+1} \quad \text{if} \quad j = l \text{ and } i = k$$
$$= -d_{j+1,l+1} \quad \text{if} \quad j \neq l \text{ and } i = k \tag{37c}$$

$$C_{j,l}^{(NZ+1,k)} = \hat{G}Z_{NZ+1,k}^{(1)} \quad \text{if} \quad j = l \text{ and } k \neq NZ + 1$$
$$= \hat{G}Z_{NZ+1,NZ+1}^{(1)}\big|_{\text{element }N} - \hat{G}Z_{1,1}^{(1)}\big|_{\text{element }N+1} \quad \text{if} \quad j = l \text{ and } k = NZ + 1$$
$$= 0 \quad \text{if} \quad j \neq l. \tag{37d}$$

For $\mathbf{A}^{(NE)}$:

$$C_{j,l}^{(1,1)} = C_{j,l}^{(NZ+1,NZ+1)}\big|_{\text{element }NE-1} \quad \text{if} \quad j = l$$
$$= 0 \quad \text{otherwise} \tag{38a}$$

$$C_{j,l}^{(1,k)} = -\hat{G}Z_{1,k}^{(1)} \quad \text{if} \quad j = l$$
$$= 0 \quad \text{otherwise } (k \neq 1) \tag{38b}$$

$$
\begin{aligned}
C_{j,l}^{(i,k)} &= a_{j+1,i}\hat{G}Z_{i,k}^{(1)} - b_{j+1,i}\hat{G}Z_{i,k}^{(2)} && \text{if} \quad j = l \text{ and } i \neq k \\
&= 0 && \text{if} \quad j \neq l \text{ and } i \neq k \\
&= a_{j+1,i}\hat{G}Z_{i,k}^{(1)} - b_{j+1,i}\hat{G}Z_{i,k}^{(2)} - d_{j+1,l+1} && \text{if} \quad j = l \text{ and } i = k \\
&= -d_{j+1,l+1} && \text{if} \quad j \neq l \text{ and } i = k
\end{aligned}
\tag{38c}
$$

$$
\begin{aligned}
C_{j,l}^{(NZ+1,k)} &= 0 && \text{if} \quad k \neq NZ + 1 \\
&= 1 && \text{if} \quad k = NZ + 1 \text{ and } j = l \\
&= 0 && \text{if} \quad k = NZ + 1 \text{ and } j \neq l.
\end{aligned}
\tag{38d}
$$

The vectors $\mathbf{X}^{(N)}$ and $\mathbf{B}^{(N)}$ in Eq. (35b) are defined as

$$
\mathbf{X}^{(N)} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_{NZ+1} \end{bmatrix}, \qquad N = 1,\dots, NE \tag{39a}
$$

with

$$
\mathbf{X}_i = \begin{bmatrix} T_{i,2} \\ T_{i,3} \\ \vdots \\ T_{i,NR} \end{bmatrix}, \qquad i = 1,\dots, NZ + 1 \tag{39b}
$$

and

$$
\mathbf{B}^{(N)} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_{NZ+1} \end{bmatrix}, \qquad N = 1,\dots, NE \tag{40a}
$$

with

$$
\mathbf{B}_i = \begin{bmatrix} g_{2,i} \\ g_{3,i} \\ \vdots \\ g_{NR,i} \end{bmatrix}, \qquad i = 2,\dots, NZ \tag{40b}
$$

and

$$\begin{aligned} \mathbf{B}_1 &= 0 && \text{for} \quad \mathbf{B}^{(N)}, N = 2,..., NE \\ \mathbf{B}_{NZ+1} &= 0 && \text{for} \quad \mathbf{B}^{(N)}, N = 1,..., NE-1 \\ \mathbf{B}_1 &= \mathbf{X}_1 && \text{for} \quad \mathbf{B}^{(1)} \\ \mathbf{B}_{NZ+1} &= \mathbf{X}_{NZ+1} && \text{for} \quad \mathbf{B}^{(NE)}. \end{aligned} \tag{40c}$$

Equation (34) can be solved in an efficient manner again by applying the LU factorization. The storage size required for $\mathbf{A}$ is $(NR-1)^2 \times (NZ+1)^2 \times NE$, again less than the full finite-difference approximation of comparable accuracy.

Results

Temperature profiles at the centerline for different Peclet numbers obtained by both the Chebyshev-finite difference and the Chebyshev-finite element methods, together with results from the analytical solution are presented in Fig. 1. The temperature profiles determined by the Chebyshev expansion methods agree well with those of the analytical solution [3]. Figure 2 shows temperature profiles at $r' = 0.5$ calculated by both Chebyshev expansion methods. The profiles match each other very well. However, at $r' = 0.9045$, close to the pipe wall where a singularity in temperature at $z' = 0$ exists, discrepancies between the profiles determined by the two Chebyshev expansion methods are observed (Fig. 3). Discrepancies between the predictions of

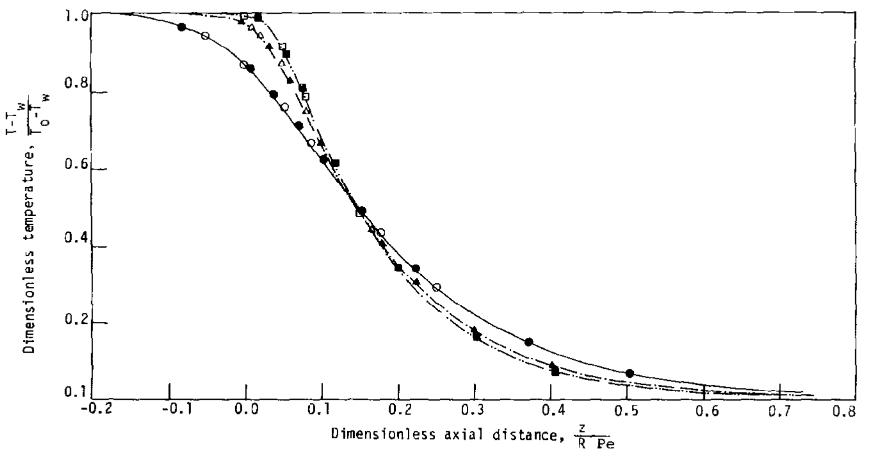


FIG. 1. Temperature profiles at radial position $r' = 0$ for various Peclet numbers. Chebyshev-finite difference methods: —, Pe = 5; —·—, Pe = 10; —··—, Pe = 20. Chebyshev-finite element method: ●, Pe = 5; ▲, Pe = 10; ■, Pe = 20. Analytical solution: ○, Pe = 5; △, Pe = 10; □, Pe = 20.

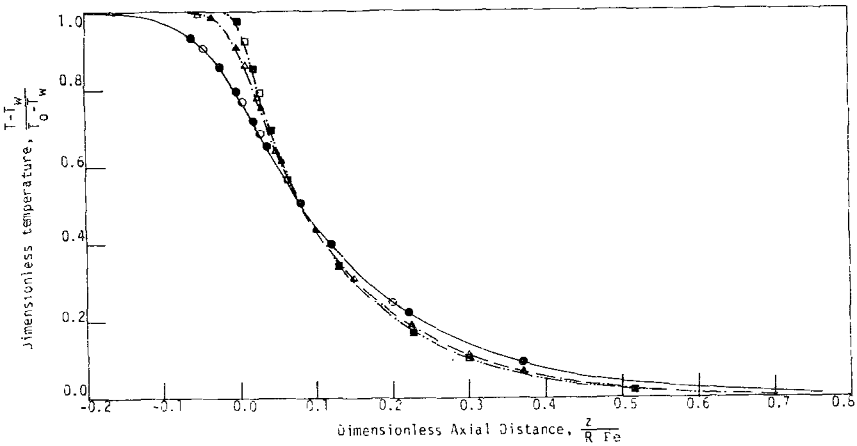


FIG. 2. Temperature profiles at radial position $r' = 0.5$ for various Peclet numbers. Chebyshev-finite difference methods: —, Pe = 5; –·–, Pe = 10; –··–, Pe = 20. Chebyshev-finite element method: ●, Pe = 5; ▲, Pe = 10; ■, Pe = 20. Analytical solution: ○, Pe = 5; △, Pe = 10; □, Pe = 20.

the two methods are also observed at $z' = 0$ where the temperature field undergoes steep changes (Table I). The Chebyshev-finite element method was proved to be more efficient in resolving singularities and non-smooth behavior in the temperature. For Pe > 100, the boundary layer character of the solution dominates and the Chebyshev-finite element method is again proven to be superior to the Chebyshev-finite difference method.

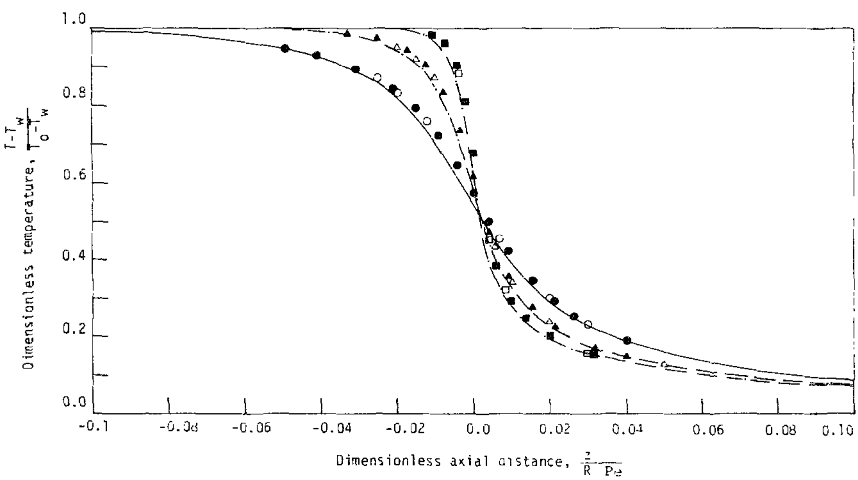


FIG. 3. Temperature profiles at radial position $r' = 0.9045$ for various Peclet numbers. Chebyshev-finite difference methods: —, Pe = 5; –·–, Pe = 10; –··–, Pe = 20. Chebyshev-finite element method: ●, Pe = 5; ▲, Pe = 10; ■, Pe = 20. Analytical solution: ○, Pe = 5; △, Pe = 10; □, Pe = 20.

TABLE I

Temperature Profiles at $z' = 0$ for Different Peclet Numbers Calculated by the Chebyshev-Finite Difference (CFD) and the Chebyshev-Finite Element (CFE) Methods

Pe	5		10		20		40	
r'	CFD	CFE	CFD	CFE	CFD	CFE	CFD	CFE
0.975	0.400	0.5197	0.3202	0.5330	0.2741	0.5507	0.2379	0.5721
0.904	0.5485	0.5740	0.5649	0.6189	0.5914	0.6791	0.6189	0.7442
0.794	0.6356	0.6493	0.7071	0.7319	0.7890	0.8209	0.8692	0.9051
0.654	0.7193	0.7269	0.8232	0.8351	0.9140	0.9253	0.9737	0.9757
0.5	0.7856	0.7913	0.9003	0.9071	0.9715	0.9754	0.9969	0.9976
0.345	0.8309	0.8349	0.9435	0.9469	0.9916	0.9924	0.9998	0.9996
0.206	0.8554	0.8591	0.9628	0.9654	0.9969	0.9974	1.0000	1.0000
0.0	0.8719	0.8645	0.9733	0.9668	0.9989	0.9931	1.0000	0.9951

In Fig. 4, local Nusselt numbers defined by

$$Nu = -\frac{2}{T_b'}\left(\frac{\partial T'}{\partial r'}\right)_{r'=1} \tag{41a}$$

$$T_b' = \frac{\int_0^1 T' u_z' r' \, dr'}{\int_0^1 u_z' r' \, dr'} \tag{41b}$$

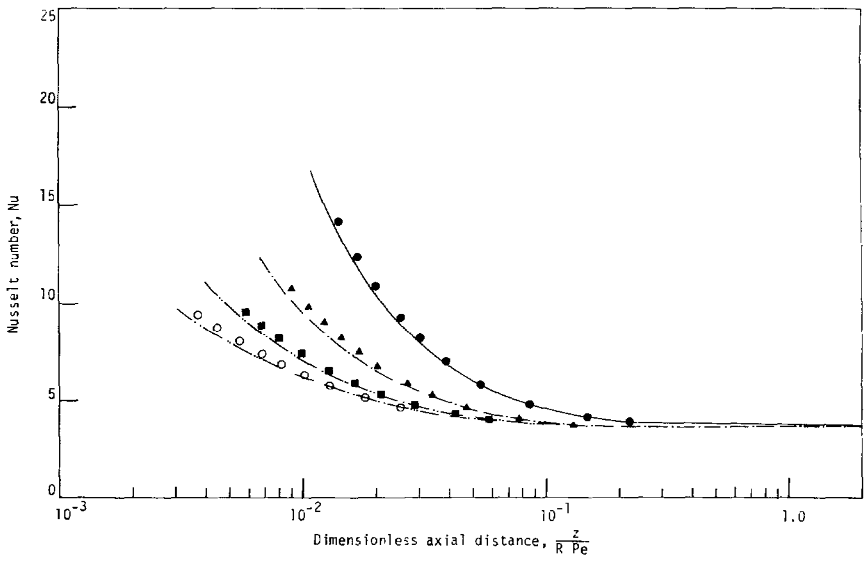


FIG. 4. Nusselt number as a function of axial distance for various Peclet numbers. Chebyshev-finite difference method: —, Pe = 5; –·–, Pe = 10; –··–, Pe = 20; –···–, Pe = 40. Chebyshev-finite element methods: ●, Pe = 5; ▲, Pe = 10; ■, Pe = 20; ○, Pe = 40.

calculated by both methods are presented. The integrals in (41b) were evaluated by the Chebyshev expansion-integration earlier described. Near the pipe entrance region, predictions by the Chebyshev-finite element method are more accurate than those from the Chebyshev-finite difference method. However, both methods predict equally well the asymptotic Nusselt number ($z \to +\infty$) in agreement with the analytical solution.

The computations described above were done on a PRIME 400 computer. Storage and computational time requirements for both Chebyshev expansion methods were particularly low.

Conclusions

The extended Graetz problem which is described by a second-order elliptic differential equation with a non-self-adjoint operator was solved by two Chebyshev expansion methods. Both these methods employ Chebyshev expansions for the derivatives in the radial direction while derivatives in the axial direction are evaluated by either finite difference approximations (Chebyshev-finite difference method) or Chebyshev expansions developed for individual sub-regions of the axial domain (Chebyshev-finite element).

In both methods the problem is reduced to a system of coupled equations which can be solved efficiently by applying LU factorization.

Both methods are applicable to general type second order elliptic problems (with self or non-self-adjoint operators, variable coefficients, etc.) and a variety of boundary conditions (homogeneous or nonhomogeneous). The methods treat the differential equation and the boundary conditions on equal footing. Both methods have low storage and computational time requirements.

The Chebyshev-finite element method proved to be superior in accuracy in resolving singularities and non-smooth behavior in the approximated functions. This is clearly demonstrated by the results of the computations for the extended Graetz problem in the neighborhoods of $z' = 0$, where heating begins, and $r' = 1$, the pipe wall. It was also proved superior to the Chebyshev-finite difference method in terms of computational economy. Half as many points in the z'-direction were required by the Chebyshev-finite element method for comparable accuracy. The Chebyshev-finite element method is similar to the global element method [10]. However, contrary to that method for which its developers [15] state that "the economics of the scheme do not appear promising," the Chebyshev-finite element method is computationally efficient and operation cost effective.

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